IMPROVED HARDY-POINCARÉ INEQUALITIES AND SHARP CARLEMAN ESTIMATES FOR DEGENERATE/SINGULAR PARABOLIC PROBLEMS

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Abstract. We consider the following class of degenerate/singular parabolic operators:

\[ Pu = u_t - (x^\alpha u)_x - \frac{\lambda}{x^\beta} u, \quad x \in (0,1), \]

associated to homogeneous boundary conditions of Dirichlet and/or Neumann type. Under optimal conditions on the parameters \( \alpha \geq 0, \beta, \lambda \in \mathbb{R} \), we derive sharp global Carleman estimates. As an application, we deduce observability and null controllability results for the corresponding evolution problem. A key step in the proof of Carleman estimates is the correct choice of the weight functions and a key ingredient in the proof takes the form of special Hardy-Poincaré inequalities.

1. Introduction. The purpose of this paper is to establish global Carleman estimates for the following class of degenerate/singular parabolic operators:

\[ Pu = u_t - (x^\alpha u)_x - \frac{\lambda}{x^\beta} u, \quad x \in (0,1), \]  \tag{1}

associated to suitable boundary conditions (that will be specified later) and where \( \alpha \geq 0, \beta, \lambda \in \mathbb{R} \) satisfy suitable assumptions (also specified later). As it is well-known, such estimates are a crucial step in view of proving null controllability properties for the corresponding evolution problem. A key step in their proof is the correct choice of the weight functions and, as we shall see, in the case of degenerate and/or singular problems, a key ingredient in the proof takes the form of special Hardy or Hardy-Poincaré inequalities. To fix the ideas, we recall here the basic form of a Hardy inequality (see for example [29] or [22, chap. 5.3]):

\[ \forall z \in H^1_0(0,1), \quad \int_0^1 z^2 \, dx \geq \frac{1}{4} \int_0^1 \frac{z^2}{x^2} \, dx. \]  \tag{2}

Theory of Carleman Estimates for uniformly parabolic operators with regular lower order terms (consider for example \( \alpha = 0 \) and \( \beta \leq 0 \) in (1)) has been largely

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developed since the works of Fursikov-Imanuvilov [27] and has been applied to many situations including for example semi-linear problems (see e.g. [25, 26, 2]).

More recently, several situations where the operator is not uniformly parabolic have been investigated. Such studies may be motivated by various physical problems (boundary layer models [9], Fisher genetics population models, Bydyko-Sellers climate models... ) where degenerate parabolic operators naturally arise. We refer to [3, 32, 6] for the study of parabolic operators coupling transport and diffusion phenomena and to [12, 1, 33] for the study of parabolic operators with degenerate diffusion at the boundary. In particular, new Carleman estimates (and consequently null controllability properties) were established in [12] for the operator

$$Pu = u_t - (x^\alpha u)_x, \quad x \in (0,1),$$

with suitable boundary conditions and under the assumption $0 \leq \alpha < 2$. Besides it has been proved that null controllability is false when $\alpha \geq 2$, see [13].

Another interesting situation that has not been largely studied is the case of parabolic operators with singular lower order terms. First results in this direction were obtained in [40] for the heat operator with singular potentials

$$Pu = u_t - u_{xx} - \frac{\lambda}{x^\beta} u, \quad x \in (0,1),$$

with Dirichlet boundary conditions. The case $\beta = 2$, we have the so-called inverse-square potential that arises for example in quantum mechanics or in linearized combustion problems [7, 8]. This potential is known to generate interesting phenomena since the works of Baras and Goldstein [4, 5]:

- Indeed global positive solutions exist (for any value of $\lambda \in \mathbb{R}$) if $\beta < 2$ whereas instantaneous and complete blow-up occurs (for any value of $\lambda$) if $\beta > 2$. Therefore, the exponent $\beta = 2$ is critical. This makes the case of inverse-square potentials particularly interesting.
- Next, when the exponent is critical i.e. when $\beta = 2$, it is the value of the parameter $\lambda$ that determines the behavior of the equation. Indeed global positive solutions exist when $\lambda \leq 1/4$ whereas instantaneous and complete blow-up occurs when $\lambda > 1/4$. The critical value $1/4$ of the parameter $\lambda$ is the optimal value of the constant in the Hardy inequality (2).

In [40], new Carleman estimates (and consequently null controllability properties) were established for (4) under the condition $\lambda \leq 1/4$. On the contrary, in the case $\lambda > 1/4$, it has been proved in [23] that null controllability is false.

In this work, we study the operator defined by (1) that couples a degenerate diffusion coefficient with a singular potential. Under suitable conditions on $\alpha$, $\beta$ and $\lambda$, we establish sharp Carleman estimates for (1). In particular, this completes and unifies the results and proofs of [12] and [40]. Let us emphasis the fact that, even in the cases of the purely degenerate operator (3) and of the purely singular operator (4), our result improves the Carleman estimates derived in [12] and [40]. In particular, in the purely degenerate case, it provides a cost of controllability that improves the one obtained in [12], see section comments in 5.2.

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The paper is organized as follows. First of all, section 2 is devoted to the improved Hardy-Poincaré inequalities that are useful to study (1). In section 3, we discuss the assumptions on the parameters $\alpha$, $\beta$ and $\lambda$. The functional setting as well as the well-posedness are discussed in section 4. Then we state our main results (i.e. Carleman estimates) in section 5. Applications to observability together with further results are given section 6. Finally sections 7-9 are devoted to the proofs.
2. Improved Hardy-Poincaré inequalities. In the case of the purely degenerate operator (3), a key ingredient in the proof of Carleman estimates in [12] relies on the following Hardy inequalities that generalizes (2): for all \( \alpha \in [0, 2) \),
\[
\int_0^1 x^{\alpha} z_x^2 \, dx \geq \frac{1-\alpha}{4} \int_0^1 \frac{z^2}{x^{2-\alpha}} \, dx,
\]
for all \( z \in C_\infty(0, 1) \) (the space of infinitely smooth functions compactly supported in \((0, 1)\)). We refer for example to [22, chap. 5]. The (optimal) constant in (5) plays a fundamental role in the sequel. We denote this constant by
\[
\lambda(\alpha) = \frac{(1-\alpha)^2}{4}.
\]

In order to prove Carleman estimates for the degenerate/singular operator (1), the Hardy inequality (5) is no more sufficient. We need the following improved Hardy-Poincaré inequalities that completes (5):

**Theorem 2.1.** Let \( \alpha \in [0, 2) \) be given. For all \( n > 0 \) and \( \gamma < 2 - \alpha \), there exists some positive constant \( C_0 = C_0(\alpha, \gamma, n) > 0 \) such that, for all \( z \in C_\infty(0, 1) \), the following inequality holds:
\[
\int_0^1 x^{\alpha} z_x^2 \, dx + C_0 \int_0^1 z^2 \, dx \geq \lambda(\alpha) \int_0^1 \frac{z^2}{x^{2-\alpha}} \, dx + n \int_0^1 \frac{z^2}{x^{2-\gamma}} \, dx.
\]

Besides \( C_0(\alpha, \gamma, n) \) is explicitly given by
\[
C_0(\alpha, \gamma, n) = (n + 1) \frac{\gamma + \alpha - 2}{2 - \alpha + \gamma} \left( \frac{4\gamma}{(2 - \alpha)^2 - \gamma^2} \right)^{\frac{2}{2-\alpha-\gamma}}.
\]

In the case \( \alpha = 0 \), the above inequality can be found in [34] and was used in [40] to derive Carleman estimates for (4). (See also [41] for another application of such inequalities to null controllability of the wave and Schrödinger equations with singular potentials). The proof of Theorem 2.1 is given later in section 7 together with the following other form of improved Hardy-Poincaré inequality that will be useful for our purpose:

**Theorem 2.2.** Let \( \alpha \in [0, 2) \) be given. For all \( \eta > 0 \), there exists some positive constant \( C = C(\alpha, \eta) > 0 \) such that, for all \( z \in C_\infty(0, 1) \), the following inequality holds:
\[
\int_0^1 x^{\alpha+\eta} z_x^2 \, dx \leq C \int_0^1 \left( x^{\alpha} z_x^2 - \lambda(\alpha) \frac{z^2}{x^{2-\alpha}} \right) \, dx.
\]

**Remark 1.** Observe that (5) ensures that, if \( \alpha \in [0, 2) \) \backslash \{1\} and if \( z \in H_{\text{loc}}^{1, \alpha}(0, 1) \) is such that \( x^{\alpha/2} z_x \in L^2(0, 1) \), then \( z/x^{(2-\alpha)/2} \) belongs to \( L^2(0, 1) \). On the contrary, in the case \( \alpha = 1 \), (5) (which trivially holds true since \( \lambda(1) = 0 \)) does not provide this information anymore. In particular, when \( \sqrt{x} z_x \in L^2(0, 1) \), (5) does not imply that \( z/\sqrt{x} \in L^2(0, 1) \) (take for example \( z \equiv 1 \)). However a “weaker” Hardy inequality is known in that case (see e.g. [22, chap. 5]):
\[
\int_0^1 x^{\alpha} z_x^2 \, dx \geq \frac{1}{4} \int_0^1 \frac{z^2}{x(\ln x)^2} \, dx,
\]
for all \( z \in C_\infty(0, 1) \). This inequality can be improved in a way that is similar to Theorem 2.1, see section 6.2.
3. Assumptions and statement of the problem. We wish to prove Carleman estimates and study the null controllability for the equation
\[ u_t - (x^\alpha u_x)_x - \frac{\lambda}{x^{\beta}} u = 0. \]  
(9)
Depending on the value of \(\alpha\), we associate to (9) some natural boundary conditions, see for instance [19, 12, 18]:
\[ u(t, 0) = 0 = u(t, 1) \text{ in the case } 0 \leq \alpha < 1, \]
(10)
\[ (x^\alpha u_x)(t, 0) = 0 = u(t, 1) \text{ in the case } \alpha \geq 1. \]
(11)
Indeed, such degenerate problem is well-posed if we work in appropriate weighted spaces which will be described later. In such spaces, the Dirichlet boundary condition makes sense when \(0 \leq \alpha < 1\). But when \(\alpha \geq 1\), the trace at the boundary at \(x = 0\) does not make sense anymore. Thus the boundary Dirichlet condition (10) at \(x = 0\) is replaced by some suitable Neumann boundary condition (11) at \(x = 0\). Finally, some initial condition \(u(0, x) = u_0(x)\) complements the system.

Before going any further, let us make precise our assumptions on the parameters.

3.1. Assumptions when \(\alpha \neq 1\). We first assume that
\[ \alpha \in [0, 2) \setminus \{1\}. \]
Indeed, we assume \(\alpha < 2\), since null controllability is known to be false for the degenerate operator (3) when \(\alpha \geq 2\), see [13]. Besides, the case \(\alpha = 1\) that is slightly peculiar is considered separately in the next section.

As recalled in the introduction, in the case \(\alpha = 0\), the critical exponent of the singular potential \(\lambda/x^\beta\) is \(\beta = 2\). This fact is a consequence of Hardy inequality (2), see [4, 5]. More generally, for a given singular potential, Cabré and Martel [10] proved that existence versus blow-up of positive solutions is connected to the existence of some Hardy inequality involving the considered potential. Therefore (5) implies that the critical exponent becomes \(\beta = 2 - \alpha\) when \(\alpha \neq 0\). This leads us to assume that \(\beta \leq 2 - \alpha\). With no loss of generality, we also assume that \(\beta > 0\).

Indeed, when \(\beta \leq 0\), then the potential is no more singular and the results easily follows from [12]. In summary, we assume
\[ 0 < \beta \leq 2 - \alpha. \]
As in the case \(\alpha = 0\), the critical value of the parameter \(\lambda\) when \(\beta = 2 - \alpha\) is given by the (optimal) constant in (5), that is \(\lambda(\alpha)\). Thus we finally assume that
\[ \lambda \leq \lambda(\alpha) \text{ when } \beta = 2 - \alpha. \]
No condition on \(\lambda\) is assumed for sub-critical exponents \(\beta\), i.e. when \(\beta < 2 - \alpha\).

3.2. Assumptions when \(\alpha = 1\). When \(\alpha = 1\), the fact that \(\sqrt{x}z_x\) belongs to \(L^2(0, 1)\) does not imply that \(z/\sqrt{x} \in L^2(0, 1)\). For this reason, the case \(\beta = 2 - \alpha = 1\) is now forbidden in equation (9). However, by (6) applied with \(\gamma = \beta\), the fact \(\sqrt{x}z\) belongs to \(L^2(0, 1)\) at least implies that \(z/x^{\beta/2} \in L^2(0, 1)\) for all \(\beta < 2 - \alpha = 1\). Thus, when \(\alpha = 1\), we will study (9) assuming that
\[ 0 < \beta < 2 - \alpha = 1, \]
(and with no condition on \(\lambda\)).
Remark 2. As mentioned above, when $\alpha = 1$, the case of a potential with critical exponent $\beta = 1$, that is a potential of the form $\lambda/x$, cannot be considered anymore. However, the weaker form (8) of (5) suggests that it can be replaced by a potential having a weaker singularity, more precisely by a potential of the form $\lambda/x(\ln x)^2$. This case is considered in section 6.2.

3.3. Summary. Let us make the point on our assumptions: in the next sections, we will study the operator

$$A_1 u := (x^\alpha u_x)_x + \frac{\lambda}{x^\beta} u, \quad (12)$$

under one of the 3 following assumptions:

- **sub-critical potentials:**
  
  $$\begin{align*}
  \alpha &\in [0, 2), & 0 < \beta < 2 - \alpha, & \text{no condition on } \lambda; \\
  \alpha &\in [0, 2) \setminus \{1\}, & \beta = 2 - \alpha, & \lambda < \lambda(\alpha); \\
  \alpha &\in [0, 2) \setminus \{1\}, & \beta = 2 - \alpha, & \lambda > \lambda(\alpha). 
  \end{align*} \quad (13)$$

- **critical potential:**
  
  $$\alpha \in [0, 2) \setminus \{1\}, \quad \beta = 2 - \alpha, \quad \lambda = \lambda(\alpha). \quad (14)$$

We separate the case where both the exponent $\beta$ and the parameter $\lambda$ are critical. In this case, the potential is called **critical** and otherwise it is called **sub-critical**. As we shall see later, the case of a critical potential requires a specific functional setting and a special care in the derivation of Carleman estimates.

3.4. Statement of the controllability problem. Let $A_1$ be the operator defined in (12) and assume that (13) or (14) holds. Let $\omega$ be a nonempty subinterval of $(0, 1)$. For $T > 0$, set $Q_T = (0, T) \times (0, 1)$ and consider the initial-boundary value problem

$$\begin{align*}
  u_t - A_1 u &= h \chi_\omega, & (t, x) &\in Q_T, \\
  u(t, 0) &= 0 = u(t, 1) & t &\in (0, T), \\
  (x^\alpha u_x)(t, 0) &= 0 = u(t, 1) & t &\in (0, T), \\
  u(0, x) &= u_0(x), & x &\in (0, 1), \quad (15)
  \end{align*}$$

where $u_0$ is given in $L^2(0, 1)$ and $h \in L^2(Q_T)$. We are interested in the null controllability in time $T > 0$ with a distributed control supported in $\omega$: for all $u_0 \in L^2(0, 1)$, does there exist $h \in L^2(Q_T)$ such that $u(t = T) \equiv 0$?

In this purpose, we derive Carleman estimates for $A_1$ in section 5. But before going any further, we describe in section 4 the functional setting in which problem (15) is well-posed.

4. Functional setting and well-posedness.

4.1. The purely degenerate case $\lambda = 0$. Let us briefly recall the functional setting used to study the purely degenerate operator (3). We denote here

$$A_0 u := (x^\alpha u_x)_x.$$

We refer the reader to [14, 12] where the above operator was studied and to [18] for a similar higher dimensional case. See also [19] for the case of an operator $(a(x)u_x)_x$ with a diffusion coefficient that vanishes at both extremities of the domain, i.e. such that $a(0) = a(1) = 0$. The above unbounded operator $A_0 : D(A_0) \subset L^2(0, 1) \rightarrow$
$L^2(0,1)$ has to be studied in appropriate weighted spaces whose definitions depend on the value of $\alpha$. Indeed a natural functional setting involves the space

$$H^{1}_{\alpha}(0,1) := \{ u \in L^2(0,1) \cap H^1_{\text{loc}}((0,1]) \mid x^{\alpha/2}u_x \in L^2(0,1) \},$$

which is a Hilbert space for the scalar product

$$\forall u, v \in H^{1}_{\alpha}(0,1), \quad (u, v)_{H^{1}_{\alpha}} = \int_{0}^{1} uv + x^{\alpha}u_x v_x.$$

For any $u \in H^{1}_{\alpha}(0,1)$, the trace of $u$ at $x = 1$ obviously makes sense which allows to consider homogeneous Dirichlet condition at $x = 1$. On the other hand, the trace of $u$ at $x = 0$ only makes sense when $0 \leq \alpha < 1$, see for example [18]. This leads us to introduce the following space $H^{1}_{\alpha,0}(0,1)$ depending on the value of $\alpha$:

**Definition 4.1.** (i) For $0 \leq \alpha < 1$, we define

$$H^{1}_{\alpha,0}(0,1) := \{ u \in H^{1}_{\alpha}(0,1) \mid u(0) = u(1) = 0 \},$$

(ii) For $1 \leq \alpha < 2$, we change the definition of $H^{1}_{\alpha,0}(0,1)$ in the following way

$$H^{1}_{\alpha,0}(0,1) := \{ u \in H^{1}_{\alpha}(0,1) \mid u(1) = 0 \}.$$

Let us mention that in both cases, $H^{1}_{\alpha,0}(0,1)$ is the closure of $C^\infty_c(0,1)$ for the norm $\| \cdot \|_{H^{1}_{\alpha}}$ (see for instance [18]). Therefore one can deduce that (5), (6), (7) and (8) also hold true for any $z \in H^{1}_{\alpha,0}(0,1)$.

Observe also that, thanks to (5) or (8), we have

$$\forall u \in H^{1}_{\alpha,0}(0,1), \quad \int_{0}^{1} u^2 \leq C_\alpha \int_{0}^{1} x^{\alpha}u_x^2,$$

for some $C_\alpha > 0$. Therefore,

$$\forall u \in H^{1}_{\alpha,0}(0,1), \quad \| u \|_{H^{1}_{\alpha,0}} := \left( \int_{0}^{1} x^{\alpha}u_x^2 \right)^{1/2},$$

defines a norm on $H^{1}_{\alpha,0}(0,1)$ that is equivalent to $\| \cdot \|_{H^{1}_{\alpha}}$.

Now we are ready to define the domain of $A_0$:

$$D(A_0) := \{ u \in H^{1}_{\alpha,0}(0,1) \cap H^2_{\text{loc}}((0,1]) \mid (x^{\alpha}u_x)_x \in L^2(0,1) \},$$

where $H^{1}_{\alpha,0}(0,1)$ is given in Definition 4.1 depending on the value of $\alpha$.

In the case $0 \leq \alpha < 1$, notice that, if $u \in D(A_0)$, then $u$ satisfies the Dirichlet boundary conditions $u(0) = u(1) = 0$.

In the case $1 \leq \alpha < 2$, one can show that $D(A_0)$ may also be characterized by

$$D(A_0) := \{ u \in L^2(0,1) \cap H^2_{\text{loc}}((0,1]) \mid x^{\alpha}u \in H^1_0(0,1), \ x^{\alpha}u_x \in H^1(0,1) \text{ and } (x^{\alpha}u_x)(0) = 0 \}.$$

We refer to [14] for the proof of the second characterization of $D(A_0)$. Notice that, if $u \in D(A_0)$, then $u$ satisfies the Neumann boundary condition $(x^{\alpha}u_x)(0) = 0$ at $x = 0$ and the Dirichlet boundary condition $u(1) = 0$ at $x = 1$. 
4.2. The degenerate/singular operator $A_1$. We assume here that (13) or (14) holds and we consider the operator

$$A_1 u = (x^\alpha u_x)_x + \frac{\lambda}{x^\beta} u.$$  

For sub-critical potentials, i.e. when (13) holds, the domain of $A_1$ will be defined as a subset of $H^{1,0}_{\alpha,0}(0,1)$. This case is treated in section 4.2.1 below. In the case of a critical potential, i.e. when (14) holds, the functional setting requires some modifications in so far as the space $H^{1,0}_{\alpha,0}(0,1)$ has to be slightly enlarged. This case is considered later in section 4.2.2.

4.2.1. The case of sub-critical potentials. In this section, we assume that (13) holds. In this case, the domain of $A_1$ is defined by

$$D(A_1) := \{ u \in H^{1,0}_{\alpha,0}(0,1) \cap H^{2,0}_{\text{loc}}((0,1]) \mid (x^\alpha u_x)_x + \frac{\lambda}{x^\beta} u \in L^2(0,1) \},$$

where $H^{1,0}_{\alpha,0}(0,1)$ is given by Definition 4.1 depending on the value of $\alpha$.

In the case $0 \leq \alpha < 1$, if $u \in D(A_1)$, then $u$ satisfies the Dirichlet boundary conditions $u(0) = u(1) = 0$.

In the case $1 \leq \alpha < 2$, we have (see section 8 for the proof):

**Proposition 1.** Assume that (13) holds with $1 \leq \alpha < 2$. Then

$$\forall u \in D(A_1), \quad x^\alpha u_x \in W^{1,1}(0,1) \text{ and } (x^\alpha u_x)(0) = 0.$$  

Therefore, in the case $1 \leq \alpha < 2$, if $u \in D(A_1)$, then $u$ satisfies the Neumann boundary condition $(x^\alpha u_x)(0) = 0$ at $x = 0$ and the Dirichlet boundary condition $u(1) = 0$ at $x = 1$.

Next, we prove (see section 8):

**Proposition 2.** Assume that (13) holds. Then there exists $k \geq 0$ and $C = C(\alpha, \beta, \lambda) > 0$ such that

$$\forall u \in H^{1,0}_{\alpha,0}(0,1), \quad \int_0^1 x^\alpha u_x^2 - \frac{\lambda u^2}{x^\beta} + ku^2 \geq C\|u\|^2_{L^2(0,1)}.$$  

*Let us mention that, when $\alpha \neq 1$ and $\lambda < \lambda(\alpha)$, $k$ is simply equal to 0.*

The above property guarantees that the bilinear form associated to $-(A_1 - kI)$ is coercive in $H^{1,0}_{\alpha,0}(0,1)$. This allows to prove (see section 8):

**Proposition 3.** Assume that (13) holds and consider the constant $k \geq 0$ that is given by Proposition 2. Then $(A_1 - kI, D(A_1))$ is a self-adjoint negative operator.

Consequently, we have the following well-posedness result (see e.g. [21, Thm. 3.1.1 and Thm. 3.2.1]):

**Theorem 4.2.** Assume that (13) holds. Consider the problem (15) with $h \equiv 0$. Then, for all $u_0 \in L^2(0,1)$, problem (15) has a unique solution

$$u \in C^0([0,T]; L^2(0,1)) \cap C^0((0,T]; D(A_1)) \cap C^1((0,T]; L^2(0,1)). \quad (16)$$

Moreover, if $u_0 \in D(A_1)$, then

$$u \in C^0([0,T]; D(A_1)) \cap C^1([0,T]; L^2(0,1)). \quad (17)$$

Consider now the non homgeneous problem (15) with $h$ be given in $L^2(Q_T).$ Then, for all $u_0 \in L^2(0,1)$, problem (15) has a unique solution

$$u \in C^0([0,T]; L^2(0,1)).$$
4.2.2. The case of a critical potential. In this section, we assume that (14) holds, i.e. \( \alpha \in [0, 2) \setminus \{1\} \), \( \beta = 2 - \alpha \) and \( \lambda = \lambda(\alpha) \). In this case, Proposition 2 is no more true. Therefore we modify the functional setting, extending the one introduced in [42] for the case \( \alpha = 0 \), \( \beta = 2 \) and \( \lambda = \lambda(0) \). Instead of \( H^\star_0(0, 1) \), we define \( H^\star_0(0, 1) \) as follows:

\[
H^\star_0(0, 1) := \{ u \in L^2(0, 1) \cap H^1_{\text{loc}}((0, 1)) \mid \int_0^1 x^{\alpha} u_x^2 - \lambda(\alpha) \frac{u^2}{x^{2-\alpha}} < +\infty \}.
\]

As it was the case for \( H^\star_0(0, 1) \), the trace at \( x = 0 \) of any \( u \in H^\star_0(0, 1) \) makes sense as soon as \( \alpha < 1 \) for the following reason:

**Proposition 4.** Let \( \alpha \in [0, 1) \). Then \( H^\star_0(0, 1) \subset W^{1,1}(0, 1) \).

The proof of Proposition 4 is given in section 8. Next we define \( H^\star_{0,0}(0, 1) \) depending on the value of \( \alpha \):

**Definition 4.3.**

(i) For \( 0 \leq \alpha < 1 \), we define \( H^\star_{\alpha,0}(0, 1) := \{ u \in H^\star_0(0, 1) \mid u(0) = u(1) = 0 \} \).

(ii) For \( 1 \leq \alpha < 2 \), we define \( H^\star_{\alpha,0}(0, 1) := \{ u \in H^\star_0(0, 1) \mid u(1) = 0 \} \).

In both cases, \( H^\star_{\alpha,0}(0, 1) \) may also be seen as the closure of \( C^\infty_c(0, 1) \) for the norm

\[
||u||_{H^\star_{\alpha,0}} := \left( \int_0^1 uv + x^{\alpha} u_x^2 - \lambda(\alpha) \frac{uv}{x^{2-\alpha}} \right)^{1/2}.
\]

Therefore one can deduce that (5), (6) and (7) also hold true for any \( z \in H^\star_{\alpha,0}(0, 1) \). Besides, thanks to (5), one can see that

\[
||u||_{H^\star_{\alpha,0}} := \left( \int_0^1 x^{\alpha} u_x^2 - \lambda(\alpha) \frac{z^2}{x^{2-\alpha}} \right)^{1/2}
\]

defines a norm on \( H^\star_{\alpha,0}(0, 1) \) that is equivalent to \( \| \cdot \|_{H^\star_{\alpha}} \).

Let us also mention that \( H^1_{\alpha,0}(0, 1) \subset \neq H^\star_{\alpha,0}(0, 1) \) and let us refer to [42] for a precise description of \( H^\star_{\alpha,0}(0, 1) \) when \( \alpha = 0 \).

In the case \( 0 \leq \alpha < 1 \), we define the domain of \( A_1 \) as follows

\[
D(A_1) := \{ u \in H^\star_{\alpha,0}(0, 1) \cap H^2_{\text{loc}}((0, 1)) \mid (x^{\alpha} u_x)_x + \frac{\lambda(\alpha)}{x^{2-\alpha}} u \in L^2(0, 1) \}.
\]

Observe that if \( u \in D(A_1) \), it satisfies \( u(0) = u(1) = 0 \).

Next, in order to define \( D(A_1) \) for \( 1 < \alpha < 2 \), we first prove (see in section 8):

**Proposition 5.** Assume that (14) holds with \( 1 < \alpha < 2 \). Then, for all \( u \in H^\star_{\alpha,0}(0, 1) \) such that \( (x^{\alpha} u_x)_x + (\lambda(\alpha)/x^{2-\alpha}) u \in L^2(0, 1) \), we have \( x^{\alpha} u_x \in W^{1,1}(0, 1) \).

Therefore, when \( 1 < \alpha < 2 \), we may define the domain of \( A_1 \) as follows:

\[
D(A_1) := \{ u \in H^\star_{\alpha,0}(0, 1) \cap H^2_{\text{loc}}((0, 1)) \mid (x^{\alpha} u_x)_x + \frac{\lambda(\alpha)}{x^{2-\alpha}} u \in L^2(0, 1) \text{ and } (x^{\alpha} u_x)(0) = 0 \}.
\]

Observe that if \( u \in D(A_1) \), it satisfies \( (x^{\alpha} u_x)(0) = u(1) = 0 \).

We are now ready to consider the question of well-posedness. Working in \( H^\star_{\alpha,0}(0, 1) \) instead of \( H^\star_{\alpha,0}(0, 1) \), Proposition 2 is now replaced by the following trivial remark:

\[
\forall u \in H^\star_{\alpha,0}(0, 1), \quad \int_0^1 x^{\alpha} z^2 - \lambda(\alpha) \frac{z^2}{x^{2-\alpha}} = \int_0^1 x^{\alpha} z^2 - \lambda(\alpha) \frac{z^2}{x^{2-\alpha}} = ||u||^2_{H^\star_{\alpha,0}}.
\]
Therefore the bilinear form associated to $-A_1$ is coercive in $H^*_{\alpha,0}(0,1)$. This allows to prove:

**Proposition 6.** Assume that (14) holds. Then $(A_1, D(A_1))$ is a self-adjoint negative operator.

We omit here the proof of Proposition 6 since it is similar to the proof of Proposition 3. As a consequence, Theorem 4.2 is still true in the present case.


5.1. Main result. Our main result consists in new Carleman estimates for the degenerate/singular problems

\[
\begin{cases}
  w_t + A_1w = f, & (t, x) \in Q_T, \\
  w(t, 0) = 0 = w(t, 1) & \text{in the case } 0 < \alpha < 1, \quad t \in (0, T), \\
  (x^\alpha w_x)(t, 0) = 0 = w(t, 1) & \text{in the case } 1 < \alpha < 2, \quad t \in (0, T), \\
  w(T, x) = w_T(x), & x \in (0, 1).
\end{cases}
\]  

(18)

More precisely, we prove:

**Theorem 5.1.** Assume that (13) or (14) holds and let $\gamma$ be given such that $0 < \gamma < 2 - \alpha$. Introduce $\sigma(t, x) := \theta(t)p(x)$ with

\[
p(x) := \frac{2 - x^{2-\alpha}}{(2 - \alpha)^2} \quad \text{and} \quad \theta(t) := \left(\frac{1}{t(T-t)}\right)^k, \quad k := 1 + \frac{2 - \alpha}{\gamma}.
\]

(i) Assume $\alpha \in (0, 2)$, $\beta = 2 - \alpha$ and $\lambda \in \mathbb{R}$. Then there exists $C > 0$ and $R_0 = R_0(2 - \alpha, \gamma, \lambda) > 0$, such that for all $R \geq R_0$, every solution $w$ of (18) satisfies

\[
\frac{R^3}{(2 - \alpha)^3} \int_{Q_T} \theta^3 x^{2-\alpha} w^2 e^{-2R\sigma} + R \int_{Q_T} \theta x^\alpha w_x^2 e^{-2R\sigma} + (1 - \alpha)^2 R \int_{Q_T} \theta w^2 e^{-2R\sigma} + R \int_{Q_T} \theta w^2 e^{-2R\sigma} \leq C \int_{Q_T} f^2 e^{-2R\sigma} + CR \int_0^T \theta(w^2 e^{-2R\sigma})_{|x=1}. \quad (19)
\]

(ii) Assume that $\alpha \in [0, 2) \setminus \{1\}$, $\beta = 2 - \alpha$ and $\lambda \leq \lambda(\alpha)$. Then there exists $C > 0$ and $R_0 = R_0(2 - \alpha, \gamma, \lambda) > 0$, such that for all $R \geq R_0$, every solution $w$ of (18) satisfies

\[
\frac{R^3}{(2 - \alpha)^3} \int_{Q_T} \theta^3 x^{2-\alpha} w^2 e^{-2R\sigma} + R \int_{Q_T} \theta x^\alpha w_x^2 - \lambda w^2 \frac{w^2}{x^{2-\alpha}} e^{-2R\sigma} + R \int_{Q_T} \theta w^2 e^{-2R\sigma} \leq C \int_{Q_T} f^2 e^{-2R\sigma} + CR \int_0^T \theta(w^2 e^{-2R\sigma})_{|x=1}. \quad (20)
\]

**Remark 3.** Thanks to (5), (20) also implies

\[
\left(1 - \frac{\lambda}{\lambda(\alpha)}\right) R \int_{Q_T} \theta x^\alpha w_x^2 e^{-2R\sigma} + (\lambda(\alpha) - \lambda) R \int_{Q_T} \theta w^2 e^{-2R\sigma} \leq C \int_{Q_T} f^2 e^{-2R\sigma} + CR \int_0^T \theta(w^2 e^{-2R\sigma})_{|x=1}. \quad (21)
\]
5.2. Additional remarks and comments. Let us explain in which way Theorem 5.1 complements the results of [12] and [40] and emphasize the fact that inequalities (19)-(21) are as sharp as can be expected.

In the case of the purely degenerate operator (3) i.e. when \( \lambda = 0 \), using (19), we retrieve the estimates of

\[
\frac{R^3}{(2-\alpha)^3} \int_{Q_T} \theta^3 x^{2-\alpha} w^2 e^{-2R\sigma}, \quad R \int_{Q_T} \theta x^{\alpha} w^2 e^{-2R\sigma}
\]

that were given in [12]. We also retrieve the estimate of

\[
\frac{(1-\alpha)^2}{4} R \int_{Q_T} \theta \frac{w^2}{x^{2-\alpha}} e^{-2R\sigma}
\]

(22)

that was used in [12] to deduce the observability inequality required to prove the result of null controllability. Observe that the estimate of \( \frac{we^{R\sigma}}{x^{(2-\alpha)/2}} \) that follows from the estimate (22), is not uniform with respect to \( \alpha \) and is lost when \( \alpha = 1 \). This phenomenon comes from the technical fact that \( \alpha = 1 \) is a peculiar case in the Hardy inequality (5). For this reason, in [12], the constant in the observability inequality explodes when \( \alpha \rightarrow 1 \). Here we complement these estimates in so far as (20) provides a new estimate of the \( H^\sigma_{0,0} \)-norm of \( we^{-R\sigma} \):

\[
R \int_{Q_T} \theta \left( \frac{w^2}{x^2} \right) e^{-2R\sigma}
\]

for any \( 0 < \gamma < 2 - \alpha \). This gives an estimate of \( \frac{we^{R\sigma}}{x^{\gamma}} \) that is now uniform with respect to \( \alpha \). As we shall see later (see section 6.1), we can deduce the observability inequality from this last estimate. Therefore, the constant in the observability inequality does not explodes anymore as \( \alpha \rightarrow 1 \). This shows that the cost of the control does not explodes when \( \alpha \rightarrow 1 \) which was not prove in [12].

In the case of the purely singular operator (4) i.e. when \( \alpha = 0 \), using (20) and (21), we retrieve the estimates of

\[
R^3 \int_{Q_T} \theta^3 x^{2-\alpha} w^2 e^{-2R\sigma}, \quad R \int_{Q_T} \theta \frac{w^2}{x^3} e^{-2R\sigma},
\]

that are uniform with respect to \( \lambda \) and that were given in [40]. We also have estimates of

\[
\left( 1 - \frac{\lambda}{\lambda(0)} \right) R \int_{Q_T} \theta w^2 e^{-2R\sigma}, \quad (\lambda(0) - \lambda) R \int_{Q_T} \theta \frac{w^2}{x^2} e^{-2R\sigma}
\]

that were also given in [40]. This implies in particular some estimate of the \( H^\sigma_0(0,1) \)-norm of \( we^{-R\sigma} \) that is not uniform with respect to \( \lambda \). Indeed the constant explodes when \( \lambda \rightarrow \lambda(0) \). This is natural since, in the critical case, \( w \) does not belong to \( H^\sigma_0(0,1) \) but belongs to the space \( H^\sigma_{0,0} \) introduced in Definition 4.3. Here we complement these estimates in so far as (20) provides a new estimate of the \( H^\sigma_{0,0} \)-norm of \( we^{-R\sigma} \):

\[
R \int_{Q_T} \theta \left( \frac{w^2}{x^2} - \frac{\lambda}{\lambda(0)} \frac{w^2}{x^2} \right) e^{-2R\sigma}
\]

that is now uniform with respect to \( \lambda \). This estimate is new and was not given in [40].

In the case of the degenerate/singular operator (4) with a potential having a sub-critical exponent \( \beta < 2 - \alpha \), (19) provides the same estimates as in the purely degenerate case. All those estimates, even the estimate of the \( H^\sigma_0 \)-norm of \( we^{-R\sigma} \) are uniform with respect to the parameter \( \lambda \). This is natural since here no condition on
λ is required (there is no critical value for λ in that case and the functional setting, based on the space $H^1_0(0,1)$, is always the same for any value of λ).

In the case of the degenerate/singular operator (4) with a potential having a critical exponent $\beta = 2 - \alpha$, (20) and (21) provide the same kind of estimates as in the purely singular case. In particular, we get an estimate of the $H^1_0$-norm of $we^{-R\sigma}$ that is not uniform with respect to λ. On the contrary, we get a uniform estimate of the $H^1_0$-norm of $we^{-R\sigma}$. For the same reason as above, these are the best estimates that can be expected in this case.

6. Applications and further results.

6.1. Application to observability and controllability. As it is well-known, in order to get controllability results, we need to derive some observability inequalities for the adjoint problem

$$
\begin{align*}
&v_t + A_1v = 0, \\
&v(t,0) = 0 = v(t,1) \quad \text{in the case } 0 \leq \alpha < 1, \quad t \in (0,T), \\
&(x^\alpha v_x)(t,0) = 0 = v(t,1) \quad \text{in the case } 1 \leq \alpha < 2, \quad t \in (0,T), \\
&v(T,x) = v_T(x), \quad x \in (0,1).
\end{align*}
$$

More precisely, we need to prove:

**Theorem 6.1.** Assume that (13) or (14) holds. Let $T > 0$ be given, and let $\omega$ be a nonempty subinterval of $(0,1)$. Then there exists $C(2 - \alpha, \lambda) > 0$ such that the solutions $v$ of (23) satisfy

$$
\int_0^1 v(0,x)^2 \, dx \leq C(2 - \alpha, \lambda) \int_0^T \int_\omega v(t,x)^2 \, dx \, dt. \tag{24}
$$

Besides, in the case $\alpha \in [0,2) \setminus \{1\}$, $\beta = 2 - \alpha$ and $\lambda \leq \lambda(\alpha)$, we have $C(2 - \alpha, \lambda) = C(2 - \alpha)$.

The proof of Theorem 6.1 is given later in section 10. Then, by standard arguments (see, e.g., [27]), a null controllability result follows:

**Theorem 6.2.** Assume that (13) or (14) holds. Let $T > 0$ be given, and let $\omega$ be a nonempty subinterval of $(0,1)$. Then, for all $u_0 \in L^2(0,1)$, there exists $h \in L^2((0,T) \times \omega)$ such that the solution of (15) satisfies $u(T) \equiv 0$ in $(0,1)$.

Besides, we have the estimate

$$
\|h\|_{L^2((0,T) \times \omega)} \leq C'(2 - \alpha, \lambda)\|u_0\|_{L^2(0,1)},
$$

for some $C'(2 - \alpha, \lambda) > 0$.

**Remark 4.** Let us comment here the purely degenerate case ($\lambda = 0$) that has been considered in [12]. In [12], the observability inequality (24) was proved for any $\alpha \in [0,2)$ with some constant $\tilde{C}(\alpha)$ depending on $\alpha$. Following the computations performed in [12], one can easily see that this constant $\tilde{C}(\alpha)$ blows up as $\alpha \to 1$ and as $\alpha \to 2^-$. The fact that it blows up as $\alpha \to 2^-$ is natural. Indeed null controllability is known to be false for $\alpha \geq 2$ in the purely degenerate case, see [13]. However, there is a priori no reason that is also blows up as $\alpha \to 1$. The blowing up of $\tilde{C}(\alpha)$ in [12] is only technical. It comes from the method based on the Hardy inequality (5) that makes the case $\alpha = 1$ peculiar.
Here we avoid this difficulty by using the improved form (6) of (5). Indeed, in the case $\lambda = 0$, we can prove that the constant in the observability inequality (24) takes the form

\[ C(2 - \alpha) = C \exp \left( \frac{c}{(2 - \alpha)^{7/2}} \right). \]

We refer to the proof of Theorem 5.1 and Theorem 6.1 for the computations. Therefore $C(2 - \alpha)$ blows up as $\alpha \to 2^-$ as expected. However, the novelty here is that this constant (and consequently the cost of the controllability) is bounded in a neighborhood of $\alpha = 1$.

6.2. Further results when $\alpha = 1$.

6.2.1. Improved Hardy-Poincaré inequalities for $\alpha = 1$. As mentioned in Remark 1, (5) is of no interest when $\alpha = 1$ but may be replaced by (8). Following the ideas of Theorem 2.1, (8) can also be improved:

**Theorem 6.3.** For all $n > 0$ and $\gamma < 1$, there exists some $C_0 = C_0(\gamma,n) > 0$ such that, for all $z \in C_c^\infty(0,1)$, the following inequality holds:

\[
\int_0^1 x z(x)^2 \, dx + C_0 \int_0^1 z^2 \, dx \geq \frac{1}{4} \int_0^1 \frac{z^2}{x^\gamma} \, dx + n \int_0^1 \frac{z^2}{x^2} \, dx.
\]

The proof of Theorem 6.3 is given in section 7. Observe that the above inequality also holds true for any $z \in H^1_{\alpha,0}(0,1)$. Therefore the fact that $z \in H^1_{\alpha,0}(0,1)$ implies that $z/(\sqrt{x} \ln x) \in L^2(0,1)$.

6.2.2. New degenerate/singular operator when $\alpha = 1$. As seen before, when $\alpha = 1$, the case of a critical exponent $\beta = 2 - \alpha = 1$, that is a potential of the form $\lambda/x$, is not relevant anymore. However, the weaker form (8) of (5) suggests that it could be replaced by a potential having a weaker singularity, more precisely by a potential of the form $\lambda/(x(\ln x)^2)$. Therefore, in the case $\alpha = 1$, we may also study the following equation

\[ u_t - (xu_x)_x - \frac{\lambda}{x(\ln x)^2} u = 0. \]

By (8), the critical value of the parameter $\lambda$ is $1/4$ which leads us to assume $\lambda \leq 1/4$. Therefore, we concentrate here on the case where $A_1$ is replaced by

\[ A_2 u := (xu_x)_x + \frac{\lambda}{x(\ln x)^2} u, \]

under one the 2 following assumptions:

- sub-critical potentials: $\lambda < \frac{1}{4}$, (27)
- critical potential: $\lambda = \frac{1}{4}$. (28)

Here again, we separate the case of a critical potential (where the parameter $\lambda$ is critical) that requires a peculiar functional setting.

Thanks to Theorem 6.3, we can produce Carleman estimates for $A_2$ that are similar to those obtained in section 5 for $A_1$. The results concerning $A_2$ are given below without proof since their proofs are similar to the proofs related to $A_1$. 

6.2.3. Well-posedness for sub-critical potentials. In this section, we assume that (27) holds. In this case, the domain of $A_2$ is defined by

$$D(A_2) := \{ u \in H^1_{1,0}(0,1) \cap H^2_{\text{loc}}((0,1)) \mid (xu_x)_x + \frac{\lambda}{x(\ln x)^2} u \in L^2(0,1) \},$$

where $H^1_{1,0}(0,1)$ is given by Definition 4.1 with $\alpha = 1$. Similarly to Proposition 1, we can prove that, when $1 \leq \alpha < 2$:

$$\forall u \in D(A_2), \quad xu_x \in W^{1,1}(0,1) \text{ and } (xu_x)(0) = 0.$$

Next we prove that the bilinear form associated to $-A_2$ is coercive in $H^1_{1,0}(0,1)$. This allows to prove $(A_2, D(A_2))$ is a self-adjoint negative operator. Consequently, Theorem 4.2 holds with $A_2$ instead of $A_1$.

6.2.4. Well-posedness for the critical potential. In this section, we assume that (28) holds and we follow the lines of section 4.2.2. Here the definition of $H^1_1(0,1)$ is modified in the following way:

$$H^1_1(0,1) := \{ u \in L^2(0,1) \cap H^1_{\text{loc}}((0,1)) \mid \int_0^1 xu_x^2 - \frac{u^2}{4x(\ln x)^2} < +\infty \}.$$

Then the definition of $H^1_{1,0}(0,1)$ is unchanged:

$$H^1_{1,0}(0,1) := \{ u \in H^1_1(0,1) \mid u(1) = 0 \}.$$

Whereas the domain of $A_2$ is now

$$D(A_2) := \{ u \in H^1_{1,0}(0,1) \cap H^2_{\text{loc}}((0,1)) \mid (xu_x)_x + \frac{1}{4x(\ln x)^2} u \in L^2(0,1)$$

and $(xu_x)(0) = 0$.

In this context, one can prove that the problem is well-posed.

6.2.5. Carleman estimates for $A_2$. One can prove:

**Theorem 6.4.** Assume that (27) or (28) holds. Then, for all $0 < \gamma < 1$, there exists $R_0 = R_0(\gamma) > 0$, such that for all $R \geq R_0$, every solution $w$ of (18), with $A_2$ instead of $A_1$, satisfies the following inequality

$$R^3 \int_{Q_T} \theta^3 x w^2 e^{-2R\sigma} + R \int_{Q_T} \theta \left( xu_x^2 - \frac{u^2}{x(\ln x)^2} \right) e^{-2R\sigma}$$

$$+ R \int_{Q_T} \frac{u^2}{x^\gamma} e^{-2R\sigma} \leq \int_{Q_T} f^2 e^{-2R\sigma} + CR \int_0^T \theta(w^2 e^{-2R\sigma})_{x=1},$$

where $\sigma(t, x) := \theta(t)p(x)$ with $p(x)$ and $\theta(t)$ defined as in Theorem 5.1.

6.3. Comments and open questions. In the context of degenerate/singular operators like $A_1$ or $A_2$, several open questions still arise.

First of all, as in [12] our result easily implies a null controllability result with a boundary control applied at $x = 1$. But the case of a boundary control at $x = 0$ has still to be considered.

Next, in the case of a purely degenerate operator, the results of [12] have been extended to other situations. In [33, 1], the authors treat the case of a degenerate operator $-(a(x)u_x)_x$ with a general coefficient $a(x)$ that vanishes at $x = 0$ and/or $x = 1$, possibly with a weakly nonlinear term $f(u)$. In [11], the authors consider the case of a degenerate operator having a non-divergence form $-a(x)u_{xx}$ and with a
drift term $b(x)u_x$. We think that the results of the present paper could be extended to degenerate/singular operators in the same situations.

Contrarily to the case of uniformly parabolic operators that has by now been largely studied (see [27] among others), the study of null controllability properties of degenerate or singular operators is still at its beginning and many further directions remain to be investigated. In particular, higher dimensional problems offer a large area of new problems. We refer to [15, 16] for a first result in higher dimension for a purely degenerate operator with degeneracy occurring at the boundary of the domain and we refer to [23] for the study in higher dimension, of a purely singular operator with an inverse-square potential with singularity inside of the domain. The study of degenerate/singular operators in higher dimension has still to be done.

In the context of degenerate and/or singular parabolic equation, some other interesting issues are inverse problems. First results in this direction have been obtained in [17, 38, 39, 16].

7. Proofs of Theorems 2.1, 2.2 and 6.3.

7.1. Proof of Theorem 2.1. First observe that, for any $n > 0$ and $\gamma \leq 0$, (6) trivially follows from (5), simply taking $C_0 = n$. Then we write

$$0 \leq \int_0^1 \left( x^{\alpha/2} z_x - \frac{1 - \alpha}{2} \frac{1}{x^{(2 - \alpha)/2}} z + \frac{1}{x^{\gamma/2}} \right)^2 dx.$$  

Expanding the above inequality, we get

$$0 \leq \int_0^1 x^{\alpha} z_x^2 + \frac{(1 - \alpha)^2}{4} \int_0^1 \frac{1}{x^{2 - \alpha}} z^2 + \int_0^1 \frac{1}{x^{\gamma - 2}} - \frac{1 - \alpha}{2} \int_0^1 \frac{1}{x^{1 - \alpha}} z_x^2$$

$$+ \int_0^1 \frac{1}{x^{(\gamma - 2)/2}} (z^2)_x - (1 - \alpha) \int_0^1 \frac{1}{x^{(2 - \alpha + \gamma)/2}} z^2.$$  

Then integrations by parts lead to

$$0 \leq \int_0^1 x^{\alpha} z_x^2 + \frac{(1 - \alpha)^2}{4} \int_0^1 \frac{1}{x^{2 - \alpha}} z^2 + \int_0^1 \frac{1}{x^{\gamma - 2}} - \frac{(1 - \alpha)^2}{2} \int_0^1 \frac{1}{x^{2 - \alpha}} z^2 - \frac{2 - \alpha - \gamma}{2} \int_0^1 \frac{1}{x^{(2 - \alpha + \gamma)/2}} z^2.$$  

Adding $n \int_0^1 z^2/x^\gamma$ to both hand sides of the above inequality, we get

$$\frac{(1 - \alpha)^2}{4} \int_0^1 \frac{1}{x^{2 - \alpha}} z^2 + n \int_0^1 \frac{1}{x^{\gamma - 2}}$$

$$\leq \int_0^1 x^{\alpha} z_x^2 + \int_0^1 \left( (n + 1) - \frac{2 - \alpha - \gamma}{2} \frac{1}{x^{(2 - \alpha + \gamma)/2}} \right) \frac{1}{x^{\gamma}} z^2.$$  

In order to get (6), it remains to show that there exists $C_0 > 0$ such that

$$\int_0^1 d(x) z^2 \leq C_0 \int_0^1 z^2.$$  

(30)

where

$$d(x) := \left( (n + 1) - \frac{2 - \alpha - \gamma}{2} \frac{1}{x^{(2 - \alpha + \gamma)/2}} \right) \frac{1}{x^{\gamma}}.$$  

We have

$$d(x) \sim \frac{2 - \alpha - \gamma}{2} \frac{1}{x^{(2 - \alpha - \gamma)/2}} \frac{1}{x^{\gamma}} \leq 0.$$  

Consequently there exists \( x_0 = x_0(\alpha, \gamma, n) \in (0, 1] \) such that \( d(x) \leq 0 \) for all \( x \in (0, x_0] \). On the other hand, there exists \( C_0 = C_0(\alpha, \gamma, n) \) such that \( |d(x)| \leq C_0 \) for all \( x \in [x_0, 1] \). We conclude that

\[
\int_{x_0}^{1} d(x) z^2 \leq \int_{x_0}^{1} d(x) z^2 \leq \int_{x_0}^{1} |d(x)| z^2 \leq C_0 \int_{0}^{1} z^2.
\]

Therefore (30) holds which concludes the proof of Theorem 2.1. The explicit expression of \( C_0 \) given in Theorem 2.1 is obtained by computing \( C_0 := \max_{[0,1]} d(x) \). \( \square \)

7.2. Proof of Theorem 2.2. Let \( \eta > 0 \) be given and let us write

\[
0 \leq \int_{0}^{1} \left( x^{\alpha/2} z_x - \frac{1 - \alpha}{2} \frac{1}{x^{(2 - \alpha)/2}} z - x^{\alpha/2 + \eta} z_x \right)^2 dx.
\]

Expanding the above inequality, we get

\[
0 \leq \int_{0}^{1} x^{\alpha} z_x^2 + \frac{(1 - \alpha)^2}{4} \int_{0}^{1} \frac{1}{x^{2 - \alpha}} z^2 + \int_{0}^{1} x^{\alpha + 2 \eta} z_x^2 - \frac{1 - \alpha}{2} \int_{0}^{1} \frac{1}{x^{1 - \alpha}} (z^2)_x
\]

\[
- 2 \int_{0}^{1} x^{\alpha + \eta} z_x^2 + \frac{(1 - \alpha)}{2} \int_{0}^{1} \frac{1}{x^{1 - \alpha - \eta}} (z^2)_x.
\]

Then integrations by parts lead to

\[
0 \leq \int_{0}^{1} x^{\alpha} z_x^2 - \lambda(\alpha) \frac{1}{x^{2 - \alpha}} z^2 + \int_{0}^{1} x^{\alpha + 2 \eta} z_x^2 \frac{1}{2} \int_{0}^{1} \frac{1}{x^{1 - \alpha}} (z^2)_x
\]

\[
+ \frac{(1 - \alpha)(1 - \alpha - \eta)}{2} \int_{0}^{1} \frac{1}{x^{2 - \alpha - \eta}} z^2.
\]

Since \( 2 - \alpha - \eta < 2 - \alpha \), we can use (5) or (8) to bound the last term and we deduce

\[
2 \int_{0}^{1} x^{\alpha + \eta} z_x^2 \leq C \int_{0}^{1} x^{\alpha} z_x^2 - \lambda(\alpha) \frac{1}{x^{2 - \alpha}} z^2 + \int_{0}^{1} x^{\alpha + 2 \eta} z_x^2,
\]

for some \( C = C(\alpha, \eta) > 0 \). Since \( x^{\alpha + 2 \eta} \leq x^{\alpha + \eta} \), the result follows:

\[
\int_{0}^{1} x^{\alpha + \eta} z_x^2 \leq C \int_{0}^{1} x^{\alpha} z_x^2 - \lambda(\alpha) \frac{1}{x^{2 - \alpha}} z^2.
\]

\( \square \)

7.3. Proof of Theorem 6.3. As for Theorem 2.1, we only consider the case \( n > 0 \) and \( 0 < \gamma < 1 \). Next we write

\[
0 \leq \int_{0}^{1} \left( \sqrt{x} z_x - \frac{1}{2} \frac{1}{\sqrt{x \ln x}} z + \frac{1}{x^{\gamma/2}} z \right)^2 dx.
\]

Expanding the above inequality, we get

\[
0 \leq \int_{0}^{1} x z_x^2 + \frac{1}{4} \int_{0}^{1} \frac{1}{x (\ln x)^2} z^2 + \int_{0}^{1} \frac{1}{x^{\gamma}} z^2 - \frac{1}{2} \int_{0}^{1} \frac{1}{\ln x} (z^2)_x
\]

\[
+ \int_{0}^{1} x^{(1-\gamma)/2} (z^2)_x - \int_{0}^{1} \frac{1}{x^{(1+\gamma)/2 \ln x}} z^2.
\]

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Then integrations by parts lead to
\[ 0 \leq \int_0^1 x^2 z^2 + \frac{1}{4} \int_0^1 \frac{1}{x(\ln x)^2} z^2 + \int_0^1 \frac{1}{x^{1+\gamma}} z^2 - \frac{1}{2} \int_0^1 \frac{1}{x(\ln x)^2} z^2 \]
\[ - \frac{1}{2} \int_0^1 \frac{1}{x^{(1+\gamma)/2}} z^2 - \int_0^1 \frac{1}{x^{(1+\gamma)/2} \ln x} z^2. \]

Adding \( n \int_0^1 z^2/x^\gamma \) to both hand sides of the above inequality, we get
\[ \frac{1}{4} \int_0^1 \frac{1}{x(\ln x)^2} z^2 + n \int_0^1 \frac{1}{x^{1+\gamma}} z^2 \]
\[ \leq \int_0^1 x^2 z^2 + \int_0^1 \left( n + 1 - \frac{1 - \gamma}{2} \right. \frac{1}{x^{1-\gamma}/2} - \left. \frac{1}{x^{(1-\gamma)/2} \ln x} \right) \frac{1}{x^\gamma} z^2. \]

As in the proof of Theorem 2.1, it remains to show (30) where \( d(x) \) is now defined by
\[ d(x) := \left( n + 1 - \frac{1 - \gamma}{2} \right) \frac{1}{x^{1-\gamma}/2} - \frac{1}{x^{(1-\gamma)/2} \ln x} \frac{1}{x^\gamma}. \]

Since
\[ d(x) \xrightarrow{\gamma \to 0} - \frac{1 - \gamma}{2} \frac{1}{x^{1-\gamma}/2 x^\gamma} \leq 0, \]
we can follow the end of the proof of Theorem 2.1. \( \square \)

8. Proof of Propositions 1-5.

8.1. Proof of Proposition 1. We assume here that (13) holds with \( 1 \leq \alpha < 2 \).
(Observe that this implies that \( \beta < 1 \)). Let us consider \( u \in A_1 \). Therefore \( u \) satisfies \( x^{\alpha/2}u_x \in L^2(0,1) \) and \( (x^\alpha u_x)_x + \lambda u/x^\beta \in L^2(0,1) \).

Let us denote \( w = x^\alpha u_x \) and let us prove that \( w \in W^{1,1}(0,1) \). Since \( x^{\alpha/2} u_x \in L^2(0,1) \), we have \( w \in L^1(0,1) \). Indeed
\[ \int_0^1 |w| = \int_0^1 |x^\alpha u_x| \leq \sqrt{\int_0^1 x^{2\alpha} u_x^2} \leq \sqrt{\int_0^1 x^\alpha u_x^2} < +\infty. \]

Next we write
\[ \int_0^1 |w_x| = \int_0^1 |(x^\alpha u_x)_x| \leq \int_0^1 \left| (x^\alpha u_x)_x + \frac{\lambda}{x^\beta} u \right| + \int_0^1 \left| \frac{\lambda}{x^\beta} u \right|. \]

Therefore
\[ \int_0^1 |w_x| \leq \sqrt{\int_0^1 \left| (x^\alpha u_x)_x + \frac{\lambda}{x^\beta} u \right|^2} + \frac{\lambda}{x^\beta} \int_0^1 |u| \]

The first integral in the right hand side of the above inequality is finite since \( (x^\alpha u_x)_x + \lambda u/x^\beta \in L^2(0,1) \). Besides one can check that \( u/x^\beta \) belongs to \( L^1(0,1) \):
\[ \int_0^1 \frac{|u|}{x^\beta} = \int_0^1 \frac{1}{x^{5/2} x^{5/2}} \leq \sqrt{\int_0^1 \frac{1}{x^\beta} \int_0^1 \frac{|u|^2}{x^\beta}}. \]

The first integral in the right hand side is finite since \( \beta < 1 \). And the second one is finite by (6) using the fact that \( x^{\alpha/2} u \in L^2(0,1) \) and the fact that \( \beta \leq 2 - \alpha \). Finally, we proved that \( w_x \in L^1(0,1) \) which shows that \( w \in W^{1,1}(0,1) \).
This implies that there exists some \( L \in \mathbb{R} \) such that \( w(x) = x^\alpha u_x(x) \rightarrow L \) as \( x \rightarrow 0^+ \). If \( L \neq 0 \), one would have
\[
x^{\alpha/2}u_x(x) \xrightarrow{x \rightarrow 0^+} \frac{L}{x^{\alpha/2}} \notin L^2(0,1),
\]
since \( \alpha \geq 1 \). This contradicts the fact that \( x^{\alpha/2}u_x \in L^2(0,1) \), thus \( L = 0 \). \hfill \Box

8.2. Proof of Proposition 2. Let us assume that (13) holds and let us distinguish 3 cases:

- **case 1:** \( \alpha \neq 1 \), \( \beta \leq 2 - \alpha \), \( \lambda < \lambda(\alpha) \),
- **case 2:** \( \alpha \neq 1 \), \( \beta < 2 - \alpha \), \( \lambda \geq \lambda(\alpha) \),
- **case 3:** \( \alpha = 1 \), \( \beta < 2 - \alpha \), \( \lambda \in \mathbb{R} \).

**First case:** we assume that \( \alpha \neq 1 \), \( \beta \leq 2 - \alpha \) and \( \lambda < \lambda(\alpha) \). Since \( \beta \leq 2 - \alpha \) and \( \lambda(\alpha) \neq 0 \), we can write
\[
\int_0^1 x^{\alpha}u_x^2 - \frac{u^2}{x^{\beta}} \geq \int_0^1 x^{\alpha}u_x^2 - \frac{u^2}{x^{2-\alpha}} \geq \left(1 - \frac{\lambda}{\lambda(\alpha)}\right) \int_0^1 x^{\alpha}u_x^2 + \frac{\lambda}{\lambda(\alpha)} \int_0^1 \left(x^{\alpha}u_x^2 - \lambda(\alpha) \frac{u^2}{x^{2-\alpha}}\right).
\]
Therefore, by (5), we get
\[
\int_0^1 x^{\alpha}u_x^2 - \frac{u^2}{x^{\beta}} \geq C_{\alpha,\lambda} \|u\|^2_{H^{1,\alpha}},
\]
where \( C_{\alpha,\lambda} = 1 - \lambda/\lambda(\alpha) > 0 \). Hence the result holds with \( k = 0 \).

**Second and third cases:** observe that in the two other cases, we have \( \beta < 2 - \alpha \). Apply (6) with \( \gamma = \beta \) and \( n = 2\lambda \) to obtain that there exists \( C_0 > 0 \) such that
\[
\int_0^1 \left(x^{\alpha}u_x^2 - 2\frac{u^2}{x^{\beta}}\right) + C_0 \int_0^1 u^2 \geq \lambda(\alpha) \int_0^1 \frac{u^2}{x^{2-\alpha}} \geq 0.
\]
Therefore we have
\[
\int_0^1 \left(x^{\alpha}u_x^2 - \frac{u^2}{x^{\beta}} + \frac{C_0}{2} u^2\right) \geq \frac{1}{2} \int_0^1 x^{\alpha}u_x^2 = \frac{1}{2} \|u\|^2_{H^{1,\alpha}}.
\]
Hence the result holds with \( k = C_0/2 \). \hfill \Box

8.3. Proof of Proposition 3. We assume that (13) holds and we consider \( k \) the constant given by Proposition 2.

(i) **A \( A_1 - kI \) is negative.** We compute
\[
\forall u \in D(A_1), \quad -(A_1u,u) = -(x^\alpha u_x + \frac{\lambda u}{x^{\beta}}, u) = \int_0^1 x^{\alpha}u_x^2 - \frac{u^2}{x^{\beta}}.
\]
Therefore, by Proposition 2, we have
\[
-(A_1 - kI)u,u) = \int_0^1 x^{\alpha}u_x^2 - \frac{u^2}{x^{\beta}} + ku^2 \geq C\|u\|^2_{H^{1,\alpha}} \geq 0,
\]
which proves the results.

(ii) **A \( A_1 - kI \) is self-adjoint.** In this purpose, it is sufficient to show that \( A_1 \) is self-adjoint, i.e. that \( (A_1^*, D(A_1^*)) = (A_1, D(A_1)) \). We recall that
\[
D(A_1^*) := \{ v \in L^2(0,1) \mid \exists C > 0, \forall u \in D(A_1), \ |(v, A_1 u)| \leq C\|u\|_{L^2(0,1)}\},
\]
and 
\[ \forall v \in D(A^*_1), \forall u \in D(A_1), \quad \langle A^*_1 v, u \rangle = \langle v, A_1 u \rangle. \]

Let us first prove that \( D(A_1) \subset D(A^*_1) \) and \( A^*_1|_{D(A_1)} = A_1 \). Let \( v \in D(A_1) \) be given and let us compute

\[ \forall u \in D(A_1), \quad \langle v, A_1 u \rangle = -\int_0^1 \left( x^{\alpha} u_x v_x - \lambda \frac{u v}{x^{\beta}} \right) = \langle A_1 v, u \rangle. \]

We deduce that \( v \in D(A^*_1) \) and that \( A^*_1 v = A_1 v \).

Next it remains to prove that \( D(A^*_1) \subset D(A_1) \). For this, we recall that \( H^1_{\alpha,0}(0,1) \) is a Hilbert space for the natural scalar product \( \langle \cdot, \cdot \rangle_{H^1_{\alpha,0}} \). By Proposition 2,

\[ \langle u, \phi \rangle_1 := \int_0^1 u \phi + x^{\alpha} u_x \phi_x - \lambda \frac{u \phi}{x^{\beta}} + ku \phi \]

defines another scalar product in \( H^1_{\alpha,0}(0,1) \) whose corresponding norm \( \| \cdot \|_1 \) is equivalent to \( \| \cdot \|_{H^1_{\alpha,0}} \). Hence \( H^1_{\alpha,0}(0,1) \) endowed with the scalar product \( \langle \cdot, \cdot \rangle_1 \) is also a Hilbert space.

Consequently, for all \( f \in L^2(0,1) \), there exists a unique \( u \in H^1_{\alpha,0}(0,1) \) such that

\[ \forall \phi \in H^1_{\alpha,0}(0,1), \quad \langle u, \phi \rangle_1 = \int_0^1 f \phi. \]

This implies that

\[ u - (x^{\alpha} u_x)_x - \lambda \frac{u}{x^{\beta}} + ku = f, \]

which yields in turn

\[ u \in D(A_1) \quad \text{and} \quad u - (A_1 - kI)u = f. \tag{31} \]

Now, let us consider \( v \in D(A^*_1) \) and let us show that \( v \) belongs to \( D(A_1) \). Introducing \( w := (A_1 - kI)^* v \in L^2(0,1) \), we have

\[ \forall \phi \in D(A_1), \quad \int_0^1 v (A_1 - kI) \phi = \int_0^1 (A_1 - kI)^* v \phi = \int_0^1 w \phi. \]

Let \( u \) be the solution of (31) with \( f = v - w \). Then \( w = v - u + (A_1 - kI)u \). Hence

\[ \int_0^1 v (A_1 - kI) \phi = \int_0^1 (v - u) \phi + \int_0^1 (A_1 - kI)u \phi = \int_0^1 (v - u) \phi + \int_0^1 u (A_1 - kI) \phi \]

since \( u \) and \( \phi \in D(A_1) \). Hence

\[ \forall \phi \in D(A_1), \quad \int_0^1 (v - u)((A_1 - kI) \phi - \phi) = 0. \]

Now let us take the solution of (31) with \( f = -(v - u) \), i.e. such that \( (A_1 - kI) \phi - \phi = v - u \), to obtain that \( v \equiv u \). Thus, \( v \in D(A_1) \). This shows that \( D(A^*_1) \subset D(A_1) \). \( \square \)

8.4. **Proof of Proposition 4.** Assume that \( \alpha \in [0, 1) \) and let \( u \) be given in \( H^1_{\alpha}(0,1) \) and let us prove that \( u \in W^{1,1}(0,1) \). Since \( u \in L^2(0,1) \), we have \( u \in L^1(0,1) \) and it remains to check that \( u_x \in L^1(0,1) \). In this purpose we write for \( \eta > 0 \)

\[ \int_0^1 |u_x| = \int_0^1 \frac{1}{x^{(\alpha + \eta)/2}} x^{(\alpha + \eta)/2} |u_x| \leq \int_0^1 \frac{1}{x^{\alpha + \eta}} \int_0^1 x^{\alpha + \eta} u_x^2. \]

The first integral is finite choosing \( \eta > 0 \) such that \( \alpha + \eta < 1 \) which is possible since \( \alpha < 1 \). The second one is finite by application of Theorem 2.2 and using the fact that \( u \in H^1_{\alpha}(0,1) \). This proves the result. \( \square \)
8.5. Proof of Proposition 5. We assume here that $1 < \alpha < 2$, $\beta = 2 - \alpha$ and $\lambda = \lambda(\alpha)$. Let us consider $u \in H_{\alpha,0}^*(0, 1)$ such that
\[
\int_0^1 \left| \frac{x^\alpha u_x}{x^{2-\alpha}} + \lambda(\alpha) \frac{u^2}{x^{2-\alpha}} \right|^2 < +\infty.
\]

We want to prove that $w := x^\alpha u_x$ belongs to $W^{1,1}(0, 1)$. First we prove that $w \in L^1(0, 1)$: using $u \in H_{\alpha,0}^*(0, 1)$ and using Theorem 2.2, we have
\[
\int_0^1 |w| = \int_0^1 |x^\alpha u_x| \leq \sqrt{\int_0^1 x^{2\alpha} u_x^2} < +\infty.
\]
Next we prove that $w_x \in L^1(0, 1)$:
\[
\int_0^1 |w_x| \leq \int_0^1 \left| \frac{x^\alpha u_x}{x^{2-\alpha}} + \frac{\lambda(\alpha)}{x^{2-\alpha}} u \right|^2 + |\lambda(\alpha)| \int_0^1 \frac{|u|}{x^{2-\alpha}}.
\]
The first integral in the right hand side of the above inequality is finite from the assumptions on $u$. Besides one can check that $u/x^{2-\alpha}$ belongs to $L^1(0, 1)$:
\[
\int_0^1 \frac{|u|}{x^{2-\alpha}} = \int_0^1 \frac{1}{x^{(2-\alpha+\varepsilon)/2}} \frac{|u|}{x^{(2-\alpha-\varepsilon)/2}} \leq \sqrt{\int_0^1 \frac{1}{x^{2-\alpha+\varepsilon}}} \int_0^1 \frac{u^2}{x^{2-\alpha-\varepsilon}}.
\]
The first integral is finite choosing $\varepsilon > 0$ such that $2 - \alpha + \varepsilon < 1$. The second one is finite using (6) with $\gamma = 2 - \alpha - \varepsilon < 2 - \alpha$. Finally, we proved that $w_x \in L^1(0, 1)$ which shows that $w \in W^{1,1}(0, 1)$.


Remark 5. Let us begin by a preliminary remark concerning the justification of the computations in the following proofs. As it is classical, it is sufficient to prove the result for the strong solutions $w$ of (18). But, in the present situation, even the strong solutions may not have enough regularity to justify all the integrations by parts in the space variable $x$. For example, in the domain of the operator, the $H^2$ regularity in $x$ is not guaranteed. Therefore, we may need to add some regularization argument to the standard procedure. This can be done by regularizing the potential, taking for example $\lambda/(x+1/n)^2$ instead of $\lambda/x^2$ in the definition of $A_1$. This defines a regularized operator $A_1^n$ whose domain is the same as in the purely degenerate case, that is to say $D(A_0)$. Therefore the strong solutions $w^n$ of the regularized problem possess all the regularity required to justify the computations (see [12]). Passing to the limit as $n \to +\infty$, we recover the result for the solutions $w$ of (18).

To simplify the presentation, we directly write the computations formally for the solutions $w$ of (18).

9.1. Outline of the proof. We proceed in several steps.

Step 1: Notations and rewriting of the problem. We consider
\[
\sigma(t, x) := \theta(t)p(x),
\]
where $p : [0, 1] \to \mathbb{R}$ is a function satisfying the following properties ($p$ will be chosen later):
\[
p \in C^0([0, 1]) \cap C^\infty((0, 1]), \quad p(x) > 0 \quad \text{for all } x \in [0, 1],
\]
and where \( \theta : (0, T) \to \mathbb{R} \) is defined by:

\[
\forall t \in (0, T), \quad \theta(t) := \left( \frac{1}{t(T-t)} \right)^k \quad \text{with } k := 1 + \frac{2-\alpha}{\gamma} > 1. \tag{33}
\]

This function satisfies:

\[
\theta(t) \to +\infty \text{ as } t \to 0^+ \text{ and } t \to T^-,
\]

and there is some \( c > 0 \) such that for all \( t \in (0, T) \),

\[
|\theta_t(t)| \leq c\theta(t)^{1+1/k} \quad \text{and } |\theta_{tt}(t)| \leq c\theta(t)^{1+2/k}. \tag{35}
\]

Next, for \( R > 0 \), we define

\[
z(t, x) = e^{-R\sigma(t,x)}w(t, x), \tag{36}
\]

where \( w \) solves (18). By the properties satisfied by \( p \) and \( \theta \), this transformation will have the effect to “kill” each term at time \( t = 0 \) and \( t = T \) in integrations by parts in time. In particular, we have

\[
\theta^2 z = 0, \quad \theta_z = 0 \quad \text{and } z_x = 0 \quad \text{at time } t = 0 \text{ and } t = T. \tag{37}
\]

Moreover \( z \) satisfies

\[
\begin{cases}
(e^{R\sigma})_t + (x^{\alpha}(e^{R\sigma})_x)_x + \frac{\lambda}{x^\beta}e^{R\sigma}z = f & \text{in } Q_T, \\
z(t, 0) = 0 = z(t, 1) & \text{in the case } 0 \leq \alpha < 1, \\
(x^{\alpha}z_x)_x(t, 0) + R(x^{\alpha}z_x)(t, 0) = 0 = z(t, 1) & \text{in the case } 1 \leq \alpha < 2.
\end{cases} \tag{38}
\]

Introducing the following self-adjoint and skew-adjoint operators

\[
P^+_R z = R\sigma z + R^2 x^{\alpha} \sigma^2 z + (x^{\alpha} z_x)_x + \frac{\lambda}{x^\beta} z, \quad P^-_R z = z_t + R(x^{\alpha} \sigma)_x z + 2Rx^{\alpha} \sigma z_x z_x,
\]

this first equation in system (38) becomes

\[
P_R z = P^+_R z + P^-_R z = fe^{-R\sigma}.
\]

Therefore, we have

\[
\|e^{-R\sigma}\|_2^2 = \|P^+_R z\|^2 + \|P^-_R z\|^2 + 2(P^+_R z, P^-_R z) \geq 2(P^+_R z, P^-_R z), \tag{39}
\]

where \( \|\cdot\| \) and \( \langle \cdot , \cdot \rangle \) respectively denote the usual norm and scalar product in \( L^2(Q_T) \).

**Step 2: Computation of the scalar product.** In order to bound from below of the quantity \( \|e^{-R\sigma}\|_2^2 \), we first compute the scalar product \( \langle P^+_R z, P^-_R z \rangle \) (see later in section 9.2 for the proof):

**Lemma 9.1.** The scalar product \( \langle P^+_R z, P^-_R z \rangle \) may be written as a sum of a distributed term \( A \) and a boundary term \( B \):

\[
\langle P^+_R z, P^-_R z \rangle = A + B,
\]

where the distributed term \( A \) is given by

\[
A = -\frac{1}{2} R \iint_{Q_T} \theta x^{2\alpha} p_z^2 - 2R^2 \iint_{Q_T} \theta x^{\alpha} p_z^2 z^2 \tag{40}
\]

\[
- R^3 \iint_{Q_T} \theta x^{2\alpha-1}(2xp_{xx} + \alpha p_x)p_z^2 z^2 - R \iint_{Q_T} \theta x^{2\alpha-1}(2xp_{xx} + \alpha p_x)z^2
\]

\[
- R \iint_{Q_T} \theta x^{\alpha}(x^\alpha p_x)_{xx} z_x + \beta \lambda R \int_{Q_T} \theta^{x^\alpha p_x}_{x^{\beta+1}} z^2,
\]
whereas the boundary term $B$ is given by

$$B = \int_0^T R \theta p_x(1) z_x^2 |_{x=1}$$

$$- \int_0^T \left( x^\alpha z_x z_t + R^2 \theta \alpha^2 \theta^2 x^2 + R^3 \theta^3 x^2 p_x^2 z^2 
+ R \theta x^\alpha(x^\alpha p_x)_x z z_t + R x^2 p_x^2 z^2 + \lambda R \theta \frac{p_x}{x^{(\alpha-1)^2}} z^2 \right) |_{x=0}. $$

**Step 3: Choice of the weight function $p$.** Let us now make precise the choice of the weight function $p$. We take here the same function $p$ that was introduced in [12] to treat the case of the purely degenerate operator (3):

$$\forall x \in [0,1], \quad p(x) := \frac{2 - x^{2-\alpha}}{(2-\alpha)^2}. \quad (42)$$

As we shall see, associated to the suitable improved Hardy-Poincaré inequalities stated in Theorem 2.1, it allows to simultaneously treat the degenerate diffusion coefficient together with the singular potential term.

Observe that $p$ belongs to $C^0([0,1]) \cap C^\infty((0,1))$, is positive on $[0,1]$ and:

$$\forall x \in (0,1], \quad p_x(x) = \frac{-x^{1-\alpha}}{2-\alpha}, \quad p_{xx}(x) = \frac{-1}{2-\alpha} x^{-\alpha}. $$

Hence

$$2 xp_{xx} + \alpha p_x = -x^{1-\alpha},$$

and

$$(x^\alpha p_x)_x = \frac{-1}{2-\alpha} \quad \text{thus} \quad (x^\alpha p_x)_{xx} = 0. $$

Using the above choice of $\theta$ and $p$ in Lemma 9.1, we deduce (see later in section 9.3 for the proof of this lemma):

**Lemma 9.2.** With $\theta$ and $p$ defined by (33) and (42), the distributed term $A$ becomes:

$$A = - \frac{2R}{(2-\alpha)^2} \int_{Q_T} \theta(1-2x^{2-\alpha} z^2) - \frac{2R^2}{(2-\alpha)^2} \int_{Q_T} \theta \alpha x^{2-\alpha} z^2 
+ \frac{R^3}{(2-\alpha)^2} \int_{Q_T} \theta x^{2-\alpha} z^2 + R \int_{Q_T} \theta x^\alpha z_x^2 - \frac{\beta R}{2-\alpha} \int_{Q_T} \theta \frac{z^2}{x^\beta}. \quad (43)$$

On the other hand, the boundary term $B$ becomes:

$$B = R \int_0^T \theta p_x(1) z_x^2 |_{x=1}. \quad (44)$$

**Step 4: Lower bound on the distributed term.** Next we produce a lower bound on the term $A$ (see later in section 9.4 for the proof):

**Lemma 9.3.** There exist a constant $R_0 > 0$ such that, for all $R \geq R_0$ we have:

- in the case $\alpha \in (0,2)$, $\beta < 2 - \alpha$ and $\lambda \in \mathbb{R}$:
Hence, proof of Lemma 9.1.

Step 5: Conclusion. It follows from Steps 1-4 that, for all $R \geq R_0$,

$$\mathbf{A} + \mathbf{B} = \langle P_+^R z, P_-^R z \rangle \leq \frac{1}{2} \| f e^{-R \sigma} \|_2^2 = \frac{1}{2} \int_{Q_T} f^2 e^{-2R \sigma}.$$ 

Hence,

$$\mathbf{A} \leq \frac{1}{2} \int_{Q_T} f^2 e^{-2R \sigma} + CR \int_0^T \theta z^2 |_{x=1}. $$

Since $z = w e^{-R \sigma}$ and $z(x = 1) = (e^{-R \sigma} w(x = 1) = (e^{-R \sigma} w(x = 1) = 0$, we easily end the proof of Theorem 5.1. \hfill \Box

9.2. Proof of Lemma 9.1. Let us write

$$\langle P_+^R z, P_-^R z \rangle = Q_1 + Q_2 + Q_3 + Q_4 + Q_5,$$

where

$$Q_1 := \langle R \sigma_t z + R^2 x^\alpha \sigma_x^2 z + (x^\alpha z_x)_x, z_t \rangle,$$

$$Q_2 := R^2 (\sigma_1 (x^\alpha \sigma_x)_x z + 2 x^\alpha \sigma_x z_x),$$

$$Q_3 := R^3 (x^\alpha \sigma_x^2 z, (x^\alpha \sigma_x)_x z + 2 x^\alpha \sigma_x z_x),$$

$$Q_4 := R^4 (x^\alpha \sigma_x)_x z + 2 x^\alpha \sigma_x z_x,$$

$$Q_5 := \langle \frac{\lambda}{x^\beta} z, z_t + R (x^\alpha \sigma_x)_x z + 2 R x^\alpha \sigma_x z_x \rangle.$$

We refer to [12] for the computations concerning $Q_1$, $Q_2$, $Q_3$ and $Q_4$. By several integrations by parts in space and in time (and using (37) in the integrations by parts in time), we get:

$$Q_1 + Q_2 + Q_3 + Q_4$$

$$= \int_0^T \left[ x^\alpha z_x z_t + R^2 x^\alpha \sigma_t \sigma_x x^2 + R^3 x^2 x^\alpha \sigma_x^3 z^2 + R x^\alpha (x^\alpha \sigma_x)_x z z_x + R x^2 \sigma_x z_x^2 \right]_0^1$$

$$+ \int_{Q_T} \left( -\frac{1}{2} R \sigma_t + 2 R^2 x^\alpha \sigma_x \sigma_x x^2 - R^3 x^2 x^\alpha \sigma_x \sigma_x \right) z^2$$

$$+ \int_{Q_T} -R x^\alpha (x^\alpha \sigma_x^2 z_x)_x - R x^\alpha (x^\alpha \sigma_x)_x z z_x.$$
Next we compute $Q_5$. Using (37), we compute

$$Q_5 = \lambda \int_{Q_T} \frac{1}{x^\beta} \left( \frac{z^2}{2} \right) + R \frac{x^\alpha \sigma_x}{x^\beta} z^2 + R \frac{x^\alpha \sigma_x}{x^\beta} (z^2) x$$

$$= \lambda R \int_{Q_T} \left( \frac{x^\alpha \sigma_x}{x^\beta} z^2 + \lambda R \int_0^T \left[ \frac{\sigma_x}{x^{\beta - \alpha}} z^2 \right]_0^1 - \lambda R \int_{Q_T} \left( \frac{x^\alpha \sigma_x}{x^\beta} z^2 \right) \right)$$

$$= \lambda R \int_0^T \left[ \frac{\sigma_x}{x^{\beta - \alpha}} z^2 \right]_0^1 + \beta \lambda R \int_{Q_T} \frac{x^\alpha \sigma_x}{x^\beta + 1} z^2.$$

Then, using the fact that $\sigma(t, x) = \theta(t)p(x)$, we deduce

$$\langle P_R^+, P_R^- \rangle = Q_1 + Q_2 + Q_3 + Q_4 + Q_5 = A + B$$

with $A$ given by (40) and with

$$B = \int_0^T \left[ x^\alpha z_x z_t + R^2 \theta \theta_t x^\alpha p_x z^2 + R^3 x^2 \beta z^2 + R \theta x^\alpha (x^\alpha p_x)_x z z_x + R \theta x^\alpha p_x z^2 + \lambda R \theta (x^\alpha p_x)_x z z_x \right] \left( \frac{\sigma_x}{x^{\beta - \alpha}} z^2 \right) \left|_{x=0} \right.$$

Since $z(t, 1) = 0$ for a.e. $t \in (0, T)$ (for any value of $\alpha \in [0, 2]$) and since $p$ is smooth in a neighborhood of $x = 1$, we get (41).

9.3. Proof of Lemma 9.2. The expression (43) of $A$ directly follows from the choice (42) of $p$. Using similarly (42), $B$ becomes

$$B = \int_0^T R \theta p_x (1) z_x^2 \left|_{x=1} \right.$$

As mentioned in Remark 5, the justification of the computations may sometimes be delicate since we work in non standard weighted spaces, specially in the critical case. For this reason, we make formal computations that may be justified by the regularization process described in Remark 5. In order to understand the computations related to $B$, it helps to replace $z$ by $z^n := e^{-R\sigma} u^n$ where $u^n$ is the solution of the regularized problem in which the potential $\lambda / x^\beta$ has been replaced by $\lambda / (x + 1/n)^\beta$.

Therefore the quantity that we actually need to compute is the following one:

$$B^n = \int_0^T R \theta p_x (1) (z^n_x)^2 \left|_{x=1} \right.$$

We recall that we consider regular solutions of the regularized problem. Thus $z^n(t)$ takes its values in $D(A_0)$. At this stage, we distinguish the two cases $0 \leq \alpha < 1$ and $1 \leq \alpha < 2$ in order to take into account the different boundary conditions satisfied by $z$ at $x = 0$. 

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First case: $0 \leq \alpha < 1$. In this case, we recall that $z^n(t, 0) = 0$ for $t \in (0, T)$. Therefore

$$B^n = \int_0^T R\theta p_x(1)(z^n_x)^2|_{x=1}$$

$$- \int_0^T \left( \frac{R}{2-\alpha} \theta x^\alpha z^n x^n - \frac{R}{2-\alpha} \theta x^{1+\alpha} (z^n_x)^2 \right) |_{x=0}. $$

Since $z^n(t) \in D(A_0)$, we have $x^\alpha z^n_x(t) \in H^1(0, 1)$. It follows that the quantity $(x^\alpha z^n_x(t)) |_{x=0}$ is finite. Using $z^n(t, 0) = 0$, we deduce that $(x^\alpha z^n_x z^n) |_{x=0} = 0$ and $(x^\alpha z^n_x z^n) |_{x=0} = 0$. Finally $(x^{1+\alpha}(z^n_x)^2) |_{x=0} = (x^{1-\alpha}(x^\alpha z^n_x)^2) |_{x=0} = 0$ since $1 - \alpha > 0$. It remains

$$B^n = R \int_0^T \theta p_x(1)(z^n_x)^2 |_{x=1}. $$

Second case: $1 \leq \alpha < 2$. In this case, we recall that

$$(x^\alpha z^n_x)(t, 0) = -R \theta(t)(x^\alpha p_x z^n)(t, 0) = \frac{R}{2-\alpha} \theta(t)xz^n(t, 0).$$

Therefore

$$B^n = \int_0^T R\theta p_x(1)(z^n_x)^2|_{x=1}$$

$$- \int_0^T \left( \frac{R}{2-\alpha} \theta x \left( \frac{(z^n_x)^2}{2} \right)_t - \frac{R^2}{2-\alpha} \theta t p_x (z^n_x)^2 - 2 \frac{R^3}{(2-\alpha)^3} \theta^3 x^{3-\alpha} (z^n_x)^2 \right.$$

$$- \frac{R^2}{(2-\alpha)^2} \theta^2 x (z^n_x)^2$$

$$- \left. \frac{\lambda R}{2-\alpha} \theta \frac{x}{(x+1/n)^2} (z^n_x)^2 \right) |_{x=0}. $$

By an integration by parts with respect to $t$, we get:

$$B^n = \int_0^T R\theta p_x(1)(z^n_x)^2|_{x=1}$$

$$+ \int_0^T \left( \frac{R}{(2-\alpha)} \theta_x (z^n_x)^2 + \frac{R^2}{2-\alpha} \theta t p_x (z^n_x)^2 + 2 \frac{R^3}{(2-\alpha)^3} \theta^3 x^{3-\alpha} (z^n_x)^2 \right.$$

$$+ \frac{R^2}{(2-\alpha)^2} \theta^2 x (z^n_x)^2 + \left. \frac{\lambda R}{2-\alpha} \theta \frac{x}{(x+1/n)^2} (z^n_x)^2 \right) |_{x=0}. $$

Next we recall the following Lemma proved in [12]:

**Lemma 9.4.** Consider $\alpha \in [1, 2)$. Then for all $z \in H^1_{\alpha, 0}(0, 1)$,

$$xz^2(x) \to 0$$

as $x \to 0^+$. Using Lemma 9.4, it remains again

$$B^n = R \int_0^T \theta p_x(1)(z^n_x)^2 |_{x=1}. $$

□
9.4. Proof of Lemma 9.3. We now want to get lower bounds for the distributed terms appearing in the scalar product \( \langle F_R^+, P_R^- \rangle \). By Lemma 9.2, we have \( A = A_1 + A_2 + A_3 + A_4 \), where
\[
A_1 := \frac{R^3}{(2 - \alpha)^2} \int_{Q_T} \theta^3 x^{2-\alpha} z^2 + R \int_{Q_T} \theta x^\alpha z_x^2,
\]
\[
A_2 := -\frac{2R}{(2 - \alpha)^2} \int_{Q_T} \theta t (2 - x^{2-\alpha}) z^2,
\]
and
\[
A_3 := -\frac{2R^2}{(2 - \alpha)^2} \int_{Q_T} \theta x^\alpha z_x^2, \quad A_4 := -\frac{\beta \lambda R}{2 - \alpha} \int_{Q_T} \theta \frac{z^2}{x^\beta}.
\]
Let us first estimate the term \( A_3 \). By (35), we know that \( |\theta t| \leq c \theta^{1+1/k} \leq c \theta^2 \) since \( k > 1 \), hence \( |\theta t| \leq c \theta^3 \), and we obtain
\[
|A_3| \leq \frac{2R^2}{(2 - \alpha)^2} \int_{Q_T} |\theta t| x^{2-\alpha} z^2 \leq \frac{cR^2}{(2 - \alpha)^2} \int_{Q_T} \theta x^\alpha z_x^2.
\]
Hence
\[
A \geq \frac{R^3 - cR^2}{(2 - \alpha)^3} \int_{Q_T} \theta^3 x^{2-\alpha} z^2 + R \int_{Q_T} \theta x^\alpha z_x^2 + A_2 + A_4.
\]

In the following, we produce estimates of the last two terms \( A_2 \) and \( A_4 \). We distinguish two cases: the case of a sub-critical exponent \( \beta < 2 - \alpha \) and the case of a critical exponent \( \beta = 2 - \alpha \).

**First case:** \( \alpha \in [0, 2) \), \( 0 < \beta < 2 - \alpha \) and \( \lambda \in \mathbb{R} \).

We want to prove the result for all \( \gamma \) satisfying \( 0 < \gamma < 2 - \alpha \). However, if the result holds true for any \( \gamma \) such that \( \beta \leq \gamma < 2 - \alpha \), then it obviously also holds true for all \( \gamma \) such that \( 0 < \gamma < 2 - \alpha \). Therefore, we consider here \( \gamma \) such that \( \beta \leq \gamma < 2 - \alpha \).

Next, we first study the term \( A_4 \). In the case \( \lambda > 0 \), we apply (6) with \( n = 2\lambda + 2 \geq 2\lambda/(2 - \alpha) + 2 \) which gives:
\[
\int_0^1 x^\alpha z_x^2 - \lambda(\alpha) \frac{z^2}{x^{2-\alpha}} \geq 2 \left( \frac{\beta \lambda}{2 - \alpha} + 1 \right) \int_0^1 \frac{z^2}{x^\gamma} - K_0 \int_0^1 z^2,
\]
where \( K_0 = K_0(2 - \alpha, \gamma, \lambda) > 0 \) is given by
\[
K_0(2 - \alpha, \gamma, \lambda) := C_0(\alpha, \gamma, 2\lambda + 2) > 0
\]
with \( C_0 \) defined in Theorem 2.1. Therefore we can write
\[
A_4 = -\frac{\beta \lambda R}{2 - \alpha} \int_{Q_T} \theta \frac{z^2}{x^\beta} \geq -\frac{\beta \lambda R}{2 - \alpha} \int_{Q_T} \theta \frac{z^2}{x^\gamma}
\]
\[
\geq \left[ \frac{(1 - \alpha)^2}{8} R \int_{Q_T} \theta \frac{z^2}{x^{2-\alpha}} - \frac{R}{2} \int_{Q_T} \theta x^\alpha z_x^2 + R \int_{Q_T} \theta \frac{z^2}{x^\gamma} - \frac{K_0 R}{2} \int_{Q_T} \theta z^2 \right].
\]

In the case \( \lambda \leq 0 \), we have
\[
A_4 = -\frac{\beta \lambda R}{2 - \alpha} \int_{Q_T} \theta \frac{z^2}{x^\beta} \geq 0.
\]

Applying (6) with \( n = 2 \), that is
\[
\frac{1}{2} \left( \int_0^1 x^\alpha z_x^2 - \lambda(\alpha) \frac{z^2}{x^{2-\alpha}} \right) \geq \frac{1}{2} \left( 2 \int_0^1 \frac{z^2}{x^\gamma} - K_0 \int_0^1 z^2 \right),
\]
(48)
we obtain the same estimate as in the case $\lambda > 0$.

It follows that

$$\mathbf{A} \geq \frac{R^3 - cR^2}{(2 - \alpha)^3} \int_{Q_T} \theta^3 x^{2-\alpha} z^2 + \frac{R}{2} \int_{Q_T} \theta x^\alpha z^2 + \frac{1 - \alpha)^2}{8} R \int_{Q_T} \theta \frac{z^2}{x^{2-\alpha}}$$

$$+ R \int_{Q_T} \theta \frac{z^2}{x^\gamma} + \mathbf{A}_2 - \frac{K_0 R}{2} \int_{Q_T} \theta z^2.$$ (49)

Next, we need to estimate the term $\mathbf{A}_2$ and the last term in the above inequality.

By (35), we have

$$|\theta_{tt}|p|_\infty \leq C\theta^{1+2/k}$$

for some $C > 0$. It follows that,

$$|\mathbf{A}_2 - \frac{K_0 R}{2} \int_{Q_T} \theta z^2| \leq RK_0' \int_{Q_T} \theta^{1+2/k} z^2,$$ (50)

for some $K_0' = K_0'(2 - \alpha, \gamma, \lambda) > 0$. At this stage, we use the special choice of $k$, that is

$$k = 1 + \frac{2 - \alpha}{\gamma}.$$ 

We set $q = k$ and $q' = k/(k-1)$ so that $q^{-1} + (q')^{-1} = 1$. Then, for all $\varepsilon > 0$, we have

$$\int_{Q_T} \theta^{1+2/k} z^2 = \int_{Q_T} \left(\frac{1}{\varepsilon} \theta^{1+2/k-1/q'} x^{\gamma/q'} z^{2/q'}\right)\left(\varepsilon \theta^{1/q'} x^{-\gamma/q'} z^{2/q'}\right)$$

$$\leq C \int_{Q_T} \theta^{(1+2/k-1/q')} x^{\gamma/q'} z^2 + \varepsilon q' \int_{Q_T} \theta \frac{z^2}{x^{\gamma}}.$$ (52)

Note that

$$q \left(1 + \frac{2}{k} - \frac{1}{q'}\right) = 3, \quad \gamma q/q' = 2 - \alpha.$$ 

Thus,

$$|\mathbf{R} \int_{Q_T} \theta^{1+2/k} z^2| \leq \frac{CR}{\varepsilon q} \int_{Q_T} \theta^3 x^{2-\alpha} z^2 + \varepsilon q' R \theta \frac{z^2}{x^{\gamma}}.$$ (51)

Putting the estimate (51) in (50) and using (49), we obtain:

$$\mathbf{A} \geq \frac{R^3 - cR^2}{(2 - \alpha)^3} \int_{Q_T} \theta^3 x^{2-\alpha} z^2 + \frac{R}{2} \int_{Q_T} \theta x^\alpha z^2 + \frac{1 - \alpha)^2}{8} R \int_{Q_T} \theta \frac{z^2}{x^{2-\alpha}} + \left(1 - \varepsilon q' K_0' \right) \int_{Q_T} \theta \frac{z^2}{x^{\gamma}}.$$ 

Taking $\varepsilon = \varepsilon(2 - \alpha, \gamma, \lambda) > 0$ such that $1 - \varepsilon q' K_0' = 1/2$, we see that there exists $R_0 = R_0(2 - \alpha, \gamma, \lambda) > 0$ such that the inequality in Lemma 9.3 holds for all $R \geq R_0$.

**Second case:** $\alpha \in [0, 2) \setminus \{1\}$, $\beta = 2 - \alpha$ and $\lambda \leq \lambda(\alpha)$.

Let us fix $\gamma$ such that $0 < \gamma < 2 - \alpha$. In the present case, we observe that

$$\mathbf{A}_4 = -\lambda R \int_{Q_T} \theta \frac{z^2}{x^{2-\alpha}}.$$ 

Therefore

$$\mathbf{A} \geq \frac{R^3 - cR^2}{(2 - \alpha)^3} \int_{Q_T} \theta^3 x^{2-\alpha} z^2 + R \int_{Q_T} \theta \left(x^\alpha z^2 - \lambda \frac{z^2}{x^{2-\alpha}}\right) + \mathbf{A}_2.$$ (52)
At this point, we apply Caccioppoli’s inequality (see [12] for details): 
\[
\int_0^1 x^{\alpha} z_x^2 - \lambda \frac{z^2}{x^{2-\alpha}} \geq \int_0^1 x^{\alpha} z_x^2 - \lambda(\alpha) \frac{z^2}{x^{2-\alpha}} \geq 2 \int_0^1 \frac{z^2}{x^n} - K_0 \int_0^1 z^2, \tag{53}
\]
where \(K_0 = K_0(2 - \alpha, \gamma) > 0\) is given by 
\[
K_0(2 - \alpha, \gamma) := C_0(\alpha, \gamma, 2) > 0
\]
with \(C_0\) defined in Theorem 2.1. We deduce 
\[
A \geq \frac{R^3 - c R^2}{(2 - \alpha)^3} \int_{Q_T} \theta^3 x^{2-\alpha} z^2 + \frac{R}{2} \int_{Q_T} \theta \left( x^{\alpha} z_x^2 - \lambda \frac{z^2}{x^{2-\alpha}} \right) + R \int_{Q_T} \theta \frac{z^2}{x^n} + A_2 - \frac{K_0 R}{2} \int_{Q_T} \theta z^2. \tag{54}
\]
Next, we need to estimate the last two terms in the above inequality. We proceed as in the previous case: by (50) and (51) we have 
\[
|A_2 - \frac{K_0 R}{2} \int_{Q_T} \theta z^2| \leq \frac{C K' R}{\varepsilon^q} \int_{Q_T} \theta^3 x^{2-\alpha} z^2 + \frac{\varepsilon q' K_0'}{2} \int_{Q_T} \theta \frac{z^2}{x^n}
\]
for some \(K'_0 = K'_0(2 - \alpha, \gamma) > 0\). Then we obtain: 
\[
A \geq \left( \frac{R^3 - c R^2}{(2 - \alpha)^3} - \frac{C K'_0 R}{\varepsilon^q} \right) \int_{Q_T} \theta^3 x^{2-\alpha} z^2 + R \frac{1}{2} \int_{Q_T} \theta \left( x^{\alpha} z_x^2 - \lambda \frac{z^2}{x^{2-\alpha}} \right) + R(1 - \varepsilon q' K'_0) \int_{Q_T} \theta \frac{z^2}{x^n}.
\]
And we conclude the proof as in the first case.

\[\Box\]

**Remark 6.** Let us consider here the purely degenerate case \(\lambda = 0\). Choosing \(\gamma = (2 - \alpha)/2\), we find that \(K_0 = K_0(2 - \alpha)\) takes the form \(K_0(2 - \alpha) = C/(2 - \alpha)^{3/2}\). Therefore easy computations lead to \(R_0(2 - \alpha, (2 - \alpha)/2, 0) = C/(2 - \alpha)^{3/2}\).

10. **Proof of Theorem 6.1.** Let us briefly show that Theorem 5.1 yields (24) and hence Theorem 6.2. For the details, we refer the reader to [12] since the argument here is similar. The idea is the following one. We consider \(\omega' = (x_0', x_1') \subset \subset \omega = (x_0, x_1)\) and a smooth cut-off function \(0 \leq \psi \leq 1\) such that \(\psi = 1\) in \((0, x_0')\) and \(\psi = 0\) in \((x_1', 1)\). We define \(w := \psi v\) where \(v\) is the solution of (23). Then \(w\) satisfies (18) with some right-hand side \(f\) explicitly given in term of \(v\) and \(v_x\) and supported in \(\omega'\). Applying Theorem 5.1 with for example \(\gamma = (2 - \alpha)/2 > 0\) and with \(R = R_0(2 - \alpha, (2 - \alpha)/2, \lambda)\), we get (since \(w_x(x = 1) = 0\): 
\[
R_0 \int_{Q_T} \theta w^2 e^{-2 R_0} \leq R_0 \int_{Q_T} \theta \frac{w^2}{x^n} e^{-2 R_0} \leq C \int_0^T \int_{\omega'} (v_x^2 + v^2) e^{-2 R_0}.
\]
At this point, we apply Caccioppoli’s inequality (see [12] for details):
\[
\int_0^T \int_{\omega'} v'_x e^{-2 R_0} \leq C \int_0^T \int_0^T v^2.
\]
Therefore
\[
R_0 \int_{Q_T} \theta w^2 e^{-2 R_0} \leq C \int_0^T \int_\omega v^2.
\]
Next we use the definition of \( \phi \) to obtain a bound for \( v \) on \((0, x_0)\) of the form
\[
\int_0^T \int_0^{x_0} \theta v^2 e^{-2R_0 \sigma} \leq \frac{C}{R_0} \int_0^T \int_\omega v^2.
\]
Using the properties of \( \theta(t) \) and \( p(x) \), we deduce
\[
\frac{2}{T} e^{-cR_0/(2-\alpha)^2} \int_0^{3T/4} \int_0^{x_0} v \leq \frac{C}{R_0} \int_0^T \int_\omega v^2.
\]
Clearly, using a similar cut-off argument, \( v \) can be estimated on \((x_1, 1)\) in the same way. In this case, we use the classical Carleman estimates since the operator \( A_1 \) is neither degenerate nor singular on \((x'_0, 1)\). Therefore we get
\[
\frac{2}{T} \int_0^{3T/4} \int_0^1 v(t,x)^2 \, dx \, dt \leq \frac{C_{cR_0/(2-\alpha)^2}}{R_0} \int_0^T \int_\omega v^2 \leq \frac{C}{R_0} \int_0^T \int_\omega v^2.
\]
We conclude as usual, using
\[
\int_0^1 v(0,x)^2 \, dx \leq \frac{2}{T} \int_0^{3T/4} \int_0^1 v(t,x)^2 \, dx \, dt \leq C(2-\alpha, \lambda) \int_0^T \int_0^1 \theta v^2 e^{-2R_0 \sigma} \, dx \, dt,
\]
where
\[
C(2-\alpha, \lambda) = Ce^{cR_0/(2-\alpha)^2}
\]
with \( R_0 = R_0(2-\alpha, (2-\alpha)/2, \lambda) \).

Proof of Remark 4. Let us consider here the purely degenerate case \( \lambda = 0 \). Recall that \( R_0(2-\alpha, (2-\alpha)/2, 0) = C/(2-\alpha)^{3/2} \) by Remark 6. Therefore we get
\[
C(2-\alpha, 0) = Ce^{c/(2-\alpha)^{7/2}}.
\]

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